HOSSAM GHANEM

(2) 7.2 The Exponential Function (A)

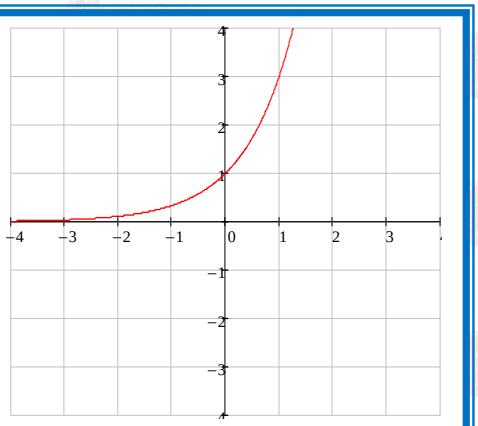
$a^3 = a \cdot a \cdot a$ $a^7 = \underbrace{a \cdot a \cdot a \dots \dots \dots . a}_{7}$ $a^{20} = \underbrace{a \cdot a \cdot a \dots \dots \dots . a}_{20}$ $a^n = \underbrace{a \cdot a \cdot a \dots \dots \dots . a}_{n}$	$2^3 = 2 \cdot 2 \cdot 2 = 8$ $\left(\frac{1}{3}\right)^4 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{81}$

$\frac{1}{a^n} = a^{-n}$	$\left(\frac{1}{5}\right)^4 = 5^{-4}$	$\frac{1}{e^x} = e^{-x}$	$\frac{1}{e^{-2x}} = e^{2x}$

$a^x \cdot a^y = a^{x+y}$	$7^9 \cdot 7^3 = 7^{12}$	$x^7 \cdot x^5 = x^{7+5} = x^{12}$	$e^{2x} \cdot e^x = e^{2x+x} = e^{3x}$

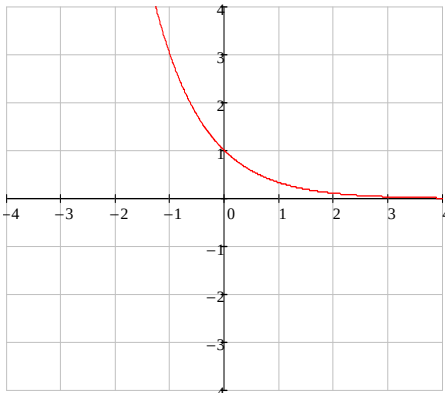
$\frac{a^x}{a^y} = a^{x-y}$	$\frac{4^9}{4^7} = 4^2$	$\frac{x^5}{x^7} = x^{5-7} = x^{-2}$	$\frac{e^{2x}}{e^x} = e^{2x-x} = e^x$

$(ab)^n = a^n \cdot b^n$	$[(x-2)(x+5)^4]^3 = (x-2)^3(x+5)^{12}$



$y = a^x \ , \ a > 1$

Graph



$y = a^x \ , 0 < a < 1$

$a > 1$	$a < 1$
$a^{\infty} \Rightarrow \infty$	$a^{\infty} \Rightarrow 0$
$a^{-\infty} \Rightarrow 0$	$a^{-\infty} \Rightarrow \infty$

Domain:

$$f(x) = e^x \quad \Rightarrow \quad D_f = \mathcal{R} \ , \quad R_f = (0, \infty)$$

Limit:

$\lim_{x\to\infty} \frac{1}{x} = 0$	$\lim_{x\to\infty} x = \infty$
$\lim_{x\to\infty} \frac{1}{x^r} = 0 \ , \ r > 0$	$\lim_{x\to\infty} x^r = \infty \ , \ r > 0$

$\lim_{x\to\infty} \frac{1}{r^x} = 0$ $\boxed{r > 1}$ $\lim_{x\to\infty} r^x = \infty$

$\lim_{x\to\infty} \frac{1}{5^x} = 0$ Ex. $\lim_{x\to\infty} 7^x = \infty$

$\lim_{x\to\infty} \left(\frac{a}{b}\right)^x = 0$ $\boxed{\begin{matrix} b > a \\ a > 0 \end{matrix}}$ $\lim_{x\to\infty} \left(\frac{b}{a}\right)^x = \infty$

$\lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0$ Ex. $\lim_{x\to\infty} \left(\frac{7}{5}\right)^x = \infty$

$\lim_{x\to\infty} e^x = \infty$	$\lim_{x\to-\infty} e^x = 0$
$\lim_{x\to\infty} e^{-x} = 0$	$\lim_{x\to-\infty} e^{-x} = \infty$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \qquad \lim_{n\to 0} (1+n)^{\frac{1}{n}} = e$$

Differential :

$\frac{d}{dx} a^x = a^x \ln a$	$\frac{d}{dx} 10^x = 10^x \ln 10$
$\frac{d}{dx} e^x = e^x$	$\frac{d}{dx} e^{x^2} = 2xe^{x^2}$

Integral:

$\int a^x \, dx = \frac{1}{\ln a} a^x + c$	$\int 7^x \, dx = \frac{1}{\ln 7} 7^x + c$
$\int e^x \, dx = e^x + c$	
$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + c$	

Example 1 find $\lim_{x\to\infty}(1.001)^x$

Solution

$\lim_{x\to\infty}(1.001)^x = \infty$ where $1.001 > 1$

Example 2 find $\lim_{x\to\infty} e^{-x^2}$

Solution

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty}\frac{1}{e^{x^2}} = 0$$

Example 3 find $\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

Solution

$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty}\frac{1-e^{-6x}}{1+e^{-6x}} = \frac{1-0}{1+0} = 1$$

Example 4 Find $\lim_{x\to-\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

Solution

$$\lim_{x\to-\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to-\infty}\frac{e^{6x}-1}{e^{6x}+1} = \frac{0-1}{0+1} = -1$$

Example 5 Find $\lim_{x\to-\infty}\frac{1-2e^x}{1+e^x}$

Solution

$$\lim_{x\to-\infty}\frac{1-2e^x}{1+e^x} = \frac{1-0}{1+0} = 1$$

Example 6 Find $\lim_{x\to\infty}\frac{1-2e^x}{1+e^x}$

Solution

$$\lim_{x\to\infty}\frac{1-2e^x}{1+e^x} = \lim_{x\to\infty}\frac{e^{-x}-2}{e^{-x}+1} = -2$$

Example 7 Find $\lim_{x\to\frac{\pi}{2}^+} e^{\tan x}$

Solution

$\lim_{x\to\frac{\pi}{2}^+}\tan x = -\infty$ $\lim_{x\to\frac{\pi}{2}^+} e^{\tan x} = 0$

Example 8 Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}$

Solution

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{5}=e^{5}$$

Example 9 Let $f(x)=7^{x}$ Find $f'(x)$

Solution

$$f'(x)=7^{x}\cdot\ln 7$$

Example 10 Let $f(x)=e^{\sin x}$ Find $f'(x)$

Solution

$$f'(x)=e^{\sin x}\cdot\cos x$$

Example 11 Let $f(x)=5^{\tan x}$ Find $f'(x)$

Solution

$$f'(x)=5^{\tan x}\cdot\ln 5\cdot\sec^{2}x$$

Example 12 Let $f(x)=e^{-3x}\cos 3x$ Find $f'(x)$

Solution

$$f'(x)=-3e^{-3x}\cos 3x-e^{-3x}\sin(3x)\cdot(3)$$

Example 13 Let $f(x)=7^{2x}\tan\sqrt{x}$ Find $f'(x)$

Solution

$$f'(x)=7^{2x}\cdot\ln 7\,(2)\tan\sqrt{x}+7^{2x}\sec^{2}\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

Example 14 Let $f(x)=\sec^{2}(e^{-4x})$ Find $f'(x)$

Solution

$$f'(x)=2\sec(e^{-4x})\sec(e^{-4x})\tan(e^{-4x})\cdot(e^{-4x})(-4)$$

Example 15 Let $f(x)=\cot(5^{2x})$ Find $f'(x)$

Solution

$$f'(x)=\csc^{2}(5^{2x})\cdot 5^{2x}\cdot\ln 5\cdot(2)$$

Example 16 Let $f(x) = \sqrt{1 + 2e^{3x}}$ Find $f'(x)$

Solution

$$f'(x) = \frac{2\,(3)e^{3x}}{2\sqrt{1 + 2e^{3x}}}$$

Example 17
11 October 1999

Use implicit differentiation to find y' if $x^4 + 2^{xy} - y^2 = 1$

Solution

$$4x^3 + 2^{xy} \cdot \ln 2\,(y + xy') - 2yy' = 0$$

$$4x^3 + 2^{xy} \cdot (\ln 2)y + 2^{xy} \cdot (\ln 2)xy' - 2yy' = 0$$

$$y'[2^{xy} \cdot (\ln 2)x - 2y] = -4x^3 - 2^{xy}\,(\ln 2)y$$

$$y' = -[4x^3 + 2^{xy}\,(\ln 2)y][2^{xy}\,(\ln 2)x - 2y]^{-1}$$

Example 18
25 April 2008

Solve the equation: $2^{3x} + 2^{5x} + 2^{7x} = 3$.

Solution

$$2^{3x} + 2^{5x} + 2^{7x} = 3$$
$$(2^x)^3 + (2^x)^5 + (2^x)^7 = 3$$

Let $2^x = y$

$$y^3 + y^5 + y^7 = 3$$

$$y = 1$$

$$2^x = 1$$

$$x = 0$$

Homework

1	Let $f(x)=e^{x^2}$ Find $f'(x)$	9	Let $f(x)=3^x-x^3$ Find $f'(x)$
2	Let $f(x)=e^{\cos x}$ Find $f'(x)$	10	Let $f(x)=7^{\cot x}$ Find $f'(x)$
3	Let $f(x)=e^{2x}\csc 3x$ Find $f'(x)$	11	Let $f(x)=\cos^2(e^{-2x})$ Find $f'(x)$
4	Let $f(x)=5^{3x}\cot\sqrt[3]{x}$ Find $f'(x)$	12	Let $f(x)=\tan(5^{3x})$ Find $f'(x)$
5	Let $f(x)=\sqrt[3]{2+e^{2x}}$ Find $f'(x)$	13	Let $f(x)=(1+3^{2x})^7$ Find $f'(x)$
6	Let $f(x)=\dfrac{e^{2x}-1}{e^{-2x}+1}$ Find $f'(x)$	14	Let $f(x)=\dfrac{3^x+1}{2^x-1}$ Find $f'(x)$
7	find $\lim\limits_{n\to\infty}(0.09)^x$	15	find $\lim\limits_{n\to-\infty}e^{-x^2}$
8	find $\lim\limits_{n\to\infty}\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$	16	find $\lim\limits_{n\to0^+}e^{\cot x}$